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Propagation property for anisotropic nonlinear diffusion equation with convection ☆

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ABSTRACT

We consider propagation property for anisotropic diffusion equation with convection in 2 dimension,

$$\partial_t(u^m) - \partial_{x_1}\left(|\partial_{x_1}u|^{p_1-1}\partial_{x_1}u\right) - \partial_{x_2}\left(|\partial_{x_2}u|^{p_2-1}\partial_{x_2}u\right) + u^{\alpha-1}\partial_{x_1}u = 0,$$

where $p_1, p_2, m, \alpha > 0$. Among the results, we show that perturbation for the nonnegative solutions propagates with infinite speed in x_1-direction and with finite speed in x_2-direction if $0 < \alpha < m < p_2$. We also show that the anisotropic propagation may appear when the convection term is weak, backward, or even missing, if $0 < p_1 \leqslant m < p_2$, $p_1 \leqslant 1$.

1. Introduction

This paper is the generalization of our previous work [10] and [11]. In [10], the propagation properties were studied for nonlinear diffusion–convection–absorption equation, including one-dimensional prototype model

$$\partial_t(u^m) - \partial_x\left(|\partial_x u|^{p-1}\partial_x u\right) - \mu|\partial_x u|^{q-1}\partial_x u + \lambda u^k = 0,$$

where m, p, q, $k > 0$ and n-dimensional simplified variant

$$\partial_t(u^m) - \Delta_{p+1}u = 0,$$

where $\Delta_{p+1}u = \operatorname{div}(|\nabla u|^{p-1}\nabla u)$ and m, $p > 0$. For the first equation we obtained complete classification of the parameters to distinguish the propagation properties. We also showed that the second equation admits finite speed propagation of perturbation if and only if $m < p$.

In [11], we studied the anisotropic propagation property for the equation

$$\partial_t(u^m) - \Delta u + u^{\alpha-1}\partial_{x_1}u = 0 \quad \text{in } \mathbb{R}^2 \times (0, T],$$

where $m, \alpha > 0$. We characterized propagation properties for nonnegative solutions of the equation in terms of m and α. For instance, in the case that $0 < \alpha < m < 1$, we find that perturbation of the solutions propagates with infinite speed in x_1-direction and with finite speed in x_2-direction. The equation with inverse convection was considered there similarly.

In the present paper, we study the following anisotropic diffusion equation

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$$\partial_t\left(u^m\right)-\partial_{x_1}\left(|\partial_{x_1}u|^{p_1-1}\partial_{x_1}u\right)-\partial_{x_2}\left(|\partial_{x_2}u|^{p_2-1}\partial_{x_2}u\right)\pm u^{\alpha-1}\partial_{x_1}u=0, \tag{1.1}$$

where $p_1,\ p_2,\ m,\ \alpha>0$.

The existence and uniqueness of solutions to the Cauchy problem for anisotropic diffusion equations could be found in [6,13,15–17]. There are many authors who studied propagation property for various parabolic equations in one-dimensional case [4,7–10,12,14,18,19,23] and multi-dimensional case [2,5,10,15,20–22,24]. See also references therein. However, as we know, few authors have studied propagation property of nonlinear anisotropic diffusion equations in the literature.

As for the notation of solutions we recall (cf. [10]):

Definition 1.1. A function $u(x_1,x_2,t)$ is called a super (sub-)solution of Eq. (1.1) in Q, where $Q=(a_1,b_1)\times(a_2,b_2)\times(0,T]$ with $-\infty\leqslant a_i<b_i\leqslant\infty$, $i=1,2$, $0<T<\infty$, if

(a) $u\in C(\overline{Q})$, $u\geqslant 0$ in Q, $\partial_{x_1}u\in L^{p_1+1}_{\mathrm{loc}}(Q)$, $\partial_{x_2}u\in L^{p_2+1}_{\mathrm{loc}}(Q)$ and
(b) in the sense of distribution

$$\partial_t\left(u^m\right)-\partial_{x_1}\left(|\partial_{x_1}u|^{p_1-1}\partial_{x_1}u\right)-\partial_{x_2}\left(|\partial_{x_2}u|^{p_2-1}\partial_{x_2}u\right)\pm u^{\alpha-1}\partial_{x_1}u\geqslant(\leqslant)\,0,$$

that is,

$$\iint_Q\left(|\partial_{x_1}u|^{p_1-1}\partial_{x_1}u\partial_{x_1}v+|\partial_{x_2}u|^{p_2-1}\partial_{x_2}u\partial_{x_2}v\right)dx\,dt\geqslant(\leqslant)\iint_Q\left(u^m\partial_t v\pm\frac{u^\alpha}{\alpha}\partial_{x_1}v\right)dx\,dt$$

for any nonnegative $v\in C_0^\infty(Q)$ satisfying $v(x_1,x_2,T)=0$.

$u(x_1,x_2,t)$ is said to be a solution if it is both a super- and sub-solution.

According to the above definition, we can prove easily the following lemma which will be used to verify various constructed solutions later.

Lemma 1.2. *Let $u\in C(\overline{Q})$ be nonnegative and $S=\{(x,t)\in Q\mid u(x,t)>0\}$. Assume that ∂S is piecewise smooth, $\partial_t(u^m)$, $D_x^2u\in C(S)$, and $\partial_{x_1}u=\partial_{x_2}u=0$ on $\partial S\cap Q$. If*

$$\partial_t\left(u^m\right)-\partial_{x_1}\left(|\partial_{x_1}u|^{p_1-1}\partial_{x_1}u\right)-\partial_{x_2}\left(|\partial_{x_2}u|^{p_2-1}\partial_{x_2}u\right)\pm u^{\alpha-1}\partial_{x_1}u\geqslant(\leqslant)\,0\quad\text{in } S,$$

then u is super (sub-)solution of the equation in Q.

In order to characterize propagation properties we make use of super- and sub-solution method. To this purpose comparison principle plays an important role.

Lemma 1.3 *(Comparison Principle). Let u and v be super- and sub-solutions of (1.1) in Q, respectively, and $u\geqslant v$ on $\overline{Q}\setminus Q$. Then $u\geqslant v$ on the whole $\overline{Q}$. Here $Q=(a_1,b_1)\times(a_2,b_2)\times(0,T]$ as above.*

With regard to the study of comparison principle see, e.g., [1,3,4,15].

The present paper is organized as follows: We consider infinite speed and finite speed propagations of perturbation for Eq. (1.1) with forward convection in Section 2. Furthermore, we study infinite and finite speed propagation for the inverse convection equation in Section 3. Finally, in Section 4 we summarize obtained results and give a classification of propagation properties in terms of the parameters p_1, p_2, m and α.

2. Equation with forward convection

In this section we consider equation

$$\partial_t\left(u^m\right)-\partial_{x_1}\left(|\partial_{x_1}u|^{p_1-1}\partial_{x_1}u\right)-\partial_{x_2}\left(|\partial_{x_2}u|^{p_2-1}\partial_{x_2}u\right)+u^{\alpha-1}\partial_{x_1}u=0, \tag{2.1}$$

where $p_1,p_2,m,\alpha>0$.

From now on we always denote $Q_T=\mathbb{R}_+\times\mathbb{R}_+\times(0,T]$, i.e.,

$$Q_T=\left\{(x_1,x_2,t)\,\middle|\,0<x_1,x_2<\infty,\ 0<t\leqslant T\right\}.$$

2.1. Infinite speed propagation

Theorem 2.1. *Let $0 < \alpha < m$, $\alpha \leqslant 1$, $\alpha < p_2$ and $u(x,t)$ be a solution of (2.1) in Q_T, satisfying $u(0, x_2^0, 0) > 0$ with some $x_2^0 > 0$. Then there exists $\tau \in (0, T]$ so that*

$$u\big(x_1, x_2^0, t\big) > 0 \quad \textit{for all } x_1 > 0,\ 0 < t < \tau.$$

This means that the perturbation propagates at infinite speed in x_1-direction. We call it Infinite Speed Propagation in x_1-direction (x_1-ISP for short).

Proof. By the continuity there exist $a \in (0, x_2^0)$, $\tau \in (0, T)$, and $\eta > 0$, so that

$$u(0, x_2, t) \geqslant \eta > 0 \quad \text{for } \big|x_2 - x_2^0\big| \leqslant a,\ 0 \leqslant t \leqslant \tau.$$

Let $Q_\tau^a = \{(x_1, x_2, t) \mid 0 < x_1 < \infty,\ |x_2 - x_2^0| < a, 0 < t \leqslant \tau\}$. We claim that $u(x_1, x_2, t) > 0$ for all $(x_1, x_2, t) \in Q_\tau^a$. The proof is as follows.

For arbitrary $x_1^0 > 0$, $t_0 \in (0, \tau)$, set $\omega = 2x_1^0/t_0$, and consider

$$v(x_1, x_2, t) = z_1(\omega t - x_1) z_2\big(x_2 - x_2^0\big), \quad (x_1, x_2, t) \in Q_\tau^a,$$

in which function $z_2(\zeta)$ is defined as

$$z_2(\zeta) = \cos\left(\frac{\pi \zeta}{2a}\right) \quad \text{for } |\zeta| \leqslant a;$$

and $z_1(\zeta)$ solves the following equation:

$$\int_0^{z_1(\zeta)} \frac{ds}{s + \epsilon s^\beta} = \zeta_+ \quad \text{for } \zeta \leqslant \zeta_0 := \int_0^\eta \frac{ds}{s + \epsilon s^\beta},$$

where $\beta \in (0, 1)$ is fixed (say $\beta = 1/2$), and $\epsilon \in (0, 1)$ will be determined later.

This gives $z_1' = z_1 + \epsilon z_1^\beta$, $0 \leqslant z_1(\zeta) \leqslant \eta$ for $\zeta \leqslant \zeta_0$,

$$z_1'' = \big(1 + \epsilon \beta z_1^{\beta - 1}\big) z_1', \qquad z_2' = -\frac{\pi}{2a} \sin\left(\frac{\pi \zeta}{2a}\right), \qquad z_2'' = -\left(\frac{\pi}{2a}\right)^2 z_2,$$
$$\partial_t v = \omega z_1' z_2, \qquad \partial_{x_1} v = -z_1' z_2, \qquad \partial_{x_2} v = z_1 z_2'.$$

Hence we have

$$\begin{aligned}
&\partial_t\big(v^m\big) - \partial_{x_1}\big(|\partial_{x_1} v|^{p_1 - 1} \partial_{x_1} v\big) - \partial_{x_2}\big(|\partial_{x_2} v|^{p_2 - 1} \partial_{x_2} v\big) + v^{\alpha - 1} \partial_{x_1} v \\
&\quad = m\omega z_1^{m-1} z_2^m z_1' - p_1 \big|z_1' z_2\big|^{p_1 - 1} z_2 z_1'' - p_2 \big|z_1 z_2'\big|^{p_2 - 1} z_1 z_2'' - z_1^{\alpha - 1} z_2^\alpha z_1' \\
&\quad \leqslant m\omega z_1^{m-1} z_2^m z_1' - p_1 \big(z_1'\big)^{p_1} z_2^{p_1} + p_2 \left(\frac{\pi}{2a}\right)^{p_2 + 1} z_1^{p_2} z_2 - z_1^{\alpha - 1} z_2^\alpha z_1' \\
&\quad \leqslant z_1^\alpha z_2^\alpha \big(m\omega z_1^{m-\alpha} z_2^{m-\alpha} - p_1 z_1^{p_1 - \alpha} z_2^{p_1 - \alpha} - 1 + C p_2 z_1^{p_2 - \alpha} z_2^{1-\alpha}\big) \\
&\quad \leqslant 0
\end{aligned}$$

for $z_1 \in (0, \eta)$, provided that $\eta > 0$ is sufficiently small, where $C = [\pi/(2a)]^{p_2 + 1}$.

In order to guarantee $0 \leqslant z_1 \leqslant \eta$, we choose $\epsilon > 0$ so small that

$$\int_0^\eta \frac{ds}{s + \epsilon s^\beta} \geqslant \omega\tau = \int_0^{z_1(\omega\tau)} \frac{ds}{s + \epsilon s^\beta},$$

and consequently, $\eta \geqslant z_1(\omega\tau) \geqslant z_1(\omega t - x_1) \geqslant 0$ in Q_τ^a.

Now we consider set

$$\begin{aligned}
S_\tau^a &= \big\{(x_1, x_2, t) \in Q_\tau^a \bigm| v(x_1, x_2, t) > 0\big\} \\
&= \big\{(x_1, x_2, t) \bigm| 0 < x_1 < \omega t,\ \big|x_2 - x_2^0\big| < a,\ 0 < t < \tau\big\}.
\end{aligned}$$

According to Lemma 1.2, to verify all conditions for v as a sub-solution in Q_τ^a we need only show zero-flux of v on $\partial S_\tau^a \cap Q_\tau^a$. This is true since

$$\partial S_\tau^a \cap Q_\tau^a = \{(x_1, x_2, t) \mid x_1 = \omega t,\ |x_2 - x_2^0| < a,\ 0 < t < \tau\},$$

and $\partial_{x_1} v = -z_1'(0) z_2(x_2 - x_2^0) = 0$, $\partial_{x_2} v = z_1(0) z_2'(x_2 - x_2^0) = 0$ when $x_1 = \omega t$.

Thus, $v(x_1, x_2, t)$ is a sub-solution of (2.1) in Q_τ^a.

In addition, we have $v \leqslant u$ in $\overline{Q_\tau^a} \setminus Q_\tau^a$. In fact,

$$v(0, x_2, t) \leqslant \eta \leqslant u(0, x_2, t) \quad \text{for } |x_2 - x_2^0| \leqslant a,\ 0 \leqslant t \leqslant \tau;$$
$$v(x_1, x_2^0 \pm a, t) = z_1(\omega t - x_1) z_2(\pm a) = 0 \leqslant u(x_1, x_2^0 \pm a, t);$$
$$v(x_1, x_2, 0) = z_1(-x_1) z_2(x_2 - x_2^0) = 0 \leqslant u(x_1, x_2, 0).$$

Therefore, by the comparison principle, $v \leqslant u$ in Q_τ^a. In particular,

$$u(x_1^0, x_2^0, t_0) \geqslant v(x_1^0, x_2^0, t_0) = z_1(\omega t_0 - x_1^0) z_2(0) > 0,$$

since $\omega t_0 - x_1^0 = x_1^0 > 0$. From the arbitrariness of $(x_1^0, x_2^0, t_0) \in Q_\tau^a$, the conclusion follows. □

Theorem 2.2. *Let $u(x, t)$ be a solution of* (2.1) *in Q_T.*

(a) *Let $p_1 \leqslant m$, $p_1 \leqslant p_2$ and $p_1 \leqslant 1$. If $u(0, x_2^0, 0) > 0$ with some $x_2^0 > 0$, then there exists $\tau \in (0, T]$ so that*

$$u(x_1, x_2^0, t) > 0 \quad \textit{for all } x_1 > 0,\ 0 < t < \tau.$$

That is, x_1-ISP property holds.

(b) *Let $p_2 \leqslant m$, $p_2 \leqslant p_1$, $p_2 \leqslant 1$ and $p_2 + 1 \leqslant \alpha$. If $u(x_1^0, 0, 0) > 0$ with some $x_1^0 > 0$, then there exists $\tau \in (0, T]$ so that*

$$u(x_1^0, x_2, t) > 0 \quad \textit{for all } x_2 > 0,\ 0 < t < \tau.$$

That is, x_2-ISP property holds.

Proof. (a) The idea of the proof is the same as previous one, except that now we take $z_1(\zeta)$ solving the equation:

$$\int_0^{z_1(\zeta)} \frac{ds}{s|\log s| + \epsilon s^\beta} = \zeta_+ \quad \text{for } \zeta \leqslant \zeta_0 := \int_0^\eta \frac{ds}{s|\log s| + \epsilon s^\beta},$$

where $\beta \in (0, 1)$ fixed and $\epsilon \in (0, 1)$ to be determined later. Then we have

$$\begin{aligned}
&z_1' = z_1|\log z_1| + \epsilon z_1^\beta, \quad z_1'' = (|\log z_1| - 1 + \epsilon\beta z_1^{\beta-1}) z_1';\\
&\partial_t(v^m) - \partial_{x_1}(|\partial_{x_1} v|^{p_1-1}\partial_{x_1} v) - \partial_{x_2}(|\partial_{x_2} v|^{p_2-1}\partial_{x_2} v) + v^{\alpha-1}\partial_{x_1} v\\
&\quad = m\omega z_1^{m-1} z_2^m z_1' - p_1|z_1' z_2|^{p_1-1} z_2 z_1'' - p_2|z_1 z_2'|^{p_2-1} z_1 z_2'' - z_1^{\alpha-1} z_2^\alpha z_1'\\
&\quad \leqslant m\omega z_1^{m-1} z_2^m z_1' - p_1|z_1'|^{p_1-1} z_2^{p_1} z_1'(|\log z_1| - 1) + p_2\left(\frac{\pi}{2a}\right)^2 z_1^{p_2}|z_2'|^{p_2-1} z_2\\
&\quad \leqslant z_1 z_2\{[m\omega z_1^{m-2} z_2^{m-1} + C p_2 z_1^{p_2-1} - p_1 z_1^{p_1-2} z_2^{p_1-1}|\log z_1|^{p_1-1}(|\log z_1| - 1)](z_1|\log z_1| + \epsilon z_1^\beta)\}\\
&\quad \leqslant 0
\end{aligned}$$

for $z_1 \in (0, \eta)$, provided that $\eta > 0$ is sufficiently small, where $C = [\pi/2a]^{p_2+1}$ as above. The rest of the proof is very similar to the previous one.

(b) The proof is almost the same as part (a). □

2.2. Finite speed propagation

Theorem 2.3. *Let $0 < m < p_1$, $m \leqslant \alpha$, and $u(x, t)$ be a solution of* (2.1) *in Q_T with initial support bounded in x_1-direction, and $u(x_1, 0, t) = 0$ for $x_1 > 0$ large enough. Then there exist $r(t) > 0$ for all $t \in [0, T]$ so that*

$$u(x_1, x_2, t) = 0 \quad \textit{for } x_1 \geqslant r(t),\ x_2 \geqslant 0,\ 0 \leqslant t \leqslant T.$$

This means that perturbation propagates with finite speed in x_1-direction. We call it Finite Speed Propagation in x_1-direction (x_1-FSP for short).

Proof. Let $a > 0$ be such that

$$u(x_1, x_2, 0) = 0 \quad \text{for } x_1 \geqslant a,\ 0 \leqslant x_2 < \infty,$$
$$u(x_1, 0, t) = 0 \quad \text{for } x_1 \geqslant a,\ 0 \leqslant t \leqslant T,$$

and $0 < M := \max u < \infty$. Then we consider $w(x_1, x_2, t) = z(\omega t - x_1 + a + \zeta_0)$ in Q_T, where function $z(\zeta)$, as well as ζ_0, is defined by

$$\int_0^{z(\zeta)} \frac{ds}{s^{m/p_1}} = \zeta_+, \qquad \zeta_0 := \int_0^M \frac{ds}{s^{m/p_1}}.$$

Hence $z' = z^{m/p_1}$, $z'' = \frac{m}{p_1} z^{\frac{m}{p_1}-1} z'$, and then

$$\begin{aligned}
&\partial_t\big(w^m\big) - \partial_{x_1}\big(|\partial_{x_1} w|^{p_1-1}\partial_{x_1} w\big) - \partial_{x_2}\big(|\partial_{x_2} w|^{p_2-1}\partial_{x_2} w\big) + w^{\alpha-1}\partial_{x_1} w \\
&\quad = mz^{m-1}\omega z' + \partial_{x_1}\big(|-z'|^{p_1-1}z'\big) - z^{\alpha-1}z' \\
&\quad = \big[mz^{m-1}\omega - mz^{m-1} - z^{\alpha-1}\big]z' \\
&\quad \geqslant 0,
\end{aligned}$$

since we choose $\omega \geqslant 1 + M^{\alpha-m}/m$. So $w(x_1, x_2, t)$ is a super-solution of (2.1) in Q_T. Noting that if $x_1 \leqslant a$, then

$$w \geqslant z(\omega t + \zeta_0) \geqslant z(\zeta_0) = M \geqslant u;$$

and if $x_1 \geqslant a$, then

$$u|_{t=0} = 0 \leqslant w|_{t=0}, \qquad u|_{x_2=0} = 0 \leqslant w|_{x_2=0}.$$

These imply that $w \geqslant u$ on $\overline{Q}_T \setminus Q_T$. By the comparison principle $w \geqslant u$ on $\overline{Q}_T$. Particularly,

$$u(x_1, x_2, t) = 0 \quad \text{for } x_1 \geqslant \omega t + a + \zeta_0,\ x_2 \geqslant 0,\ 0 \leqslant t \leqslant T,$$

which completes the proof. □

Theorem 2.4. *Let $0 < m < p_2$, and $u(x, t)$ be a solution of (2.1) in Q_T. If $u(0, x_2, t) = 0$ for $x_2 > 0$ sufficiently large, $0 < t \leqslant T$, and initial support $\operatorname{supp} u(x_1, x_2, 0)$ is bounded, then there exist $r(t) > 0$ for all $t \in (0, T]$, such that*

$$u(x_1, x_2, t) = 0, \quad x_1 \geqslant 0,\ x_2 \geqslant r(t),\ 0 < t \leqslant T.$$

Namely, Eq. (2.1) admits x_2-FSP property.

Proof. The proof follows the same line as Theorem 2.3:

We consider $w(x_1, x_2, t) = z(\omega t - x_2 + a + \zeta_0)$ in Q_T, where function $z(\zeta)$, as well as ζ_0, is defined by

$$\int_0^{z(\zeta)} \frac{ds}{s^{m/p_2}} = \zeta_+, \qquad \zeta_0 := \int_0^M \frac{ds}{s^{m/p_2}}.$$

Hence $z' = z^{m/p_2}$, $z'' = \frac{m}{p_2} z^{\frac{m}{p_2}-1} z'$, and

$$\begin{aligned}
&\partial_t\big(w^m\big) - \partial_{x_1}\big(|\partial_{x_1} w|^{p_1-1}\partial_{x_1} w\big) - \partial_{x_2}\big(|\partial_{x_2} w|^{p_2-1}\partial_{x_2} w\big) + w^{\alpha-1}\partial_{x_1} w \\
&\quad = mz^{m-1}\omega z' + \partial_{x_2}\big(|-z'|^{p_2-1}z'\big) \\
&\quad = \big[m\omega z^{m-1} - mz^{m-1}\big]z' \\
&\quad \geqslant 0,
\end{aligned}$$

provided that we choose $\omega \geqslant 1$. Therefore $w(x_1, x_2, t)$ is a super-solution of (2.1) in Q_T, and the conclusion follows. □

3. Inverse convection equation

We now consider diffusion equation with inverse convection

$$\partial_t\big(u^m\big) - \partial_{x_1}\big(|\partial_{x_1} u|^{p_1-1}\partial_{x_1} u\big) - \partial_{x_2}\big(|\partial_{x_2} u|^{p_2-1}\partial_{x_2} u\big) - u^{\alpha-1}\partial_{x_1} u = 0, \tag{3.1}$$

where $p_1, p_2, m, \alpha > 0$.

3.1. Infinite speed propagation

Theorem 3.1. *Let $u(x,t)$ be a solution of (3.1) in Q_T.*

(a) *Assume that $p_1 \leqslant m$, $p_1 \leqslant p_2$, $p_1 \leqslant 1$, $p_1 \leqslant \alpha$. If $u(0, x_2^0, 0) > 0$ with some $x_2^0 > 0$, then there exists $\tau \in (0, T]$ so that*

$$u\big(x_1, x_2^0, t\big) > 0 \quad \text{for all } x_1 > 0,\ 0 < t < \tau.$$

(b) *Assume that $p_2 \leqslant m$, $p_2 \leqslant p_1$, $p_2 \leqslant 1$, $p_2 + 1 \leqslant \alpha$. If $u(x_1^0, 0, 0) > 0$ with some $x_1^0 > 0$, then there exists $\tau \in (0, T]$ so that*

$$u\big(x_1^0, x_2, t\big) > 0 \quad \text{for all } x_2 > 0,\ 0 < t < \tau.$$

Proof. (a) By the continuity there exist $a \in (0, x_2^0)$, $\tau \in (0, T)$, and $\eta \in (0, 1)$, so that

$$u(0, x_2, t) \geqslant \eta > 0 \quad \text{for } \big|x_2 - x_2^0\big| \leqslant a,\ 0 \leqslant t \leqslant \tau.$$

Let $Q_\tau^a = \{(x_1, x_2, t) \mid 0 < x_1 < \infty, |x_2 - x_2^0| < a, 0 < t \leqslant \tau\}$. We intend to show $u(x_1, x_2, t) > 0$ for all $(x_1, x_2, t) \in Q_\tau^a$.

For arbitrary $x_1^0 > 0$, $t_0 \in (0, \tau)$, set $\omega = 2x_1^0/t_0$, and consider

$$v(x_1, x_2, t) = z_1(\omega t - x_1) z_2\big(x_2 - x_2^0\big), \quad (x_1, x_2, t) \in Q_\tau^a,$$

in which function $z_2(\zeta)$ is defined as

$$z_2(\zeta) = \cos\left(\frac{\pi\zeta}{2a}\right) \quad \text{for } |\zeta| \leqslant a;$$

and $z_1(\zeta)$ solves the following equation:

$$\int_0^{z_1(\zeta)} \frac{ds}{s|\log s| + \epsilon s^\beta} = \zeta_+ \quad \text{for } \zeta \leqslant \zeta_0 := \int_0^\eta \frac{ds}{s|\log s| + \epsilon s^\beta},$$

where $\beta \in (0, 1)$ is fixed and $\epsilon \in (0, 1)$ will be determined later.

This gives $0 \leqslant z_1(\zeta) \leqslant \eta \leqslant 1$ for $\zeta \leqslant \zeta_0$, and

$$z_1' = z_1|\log z_1| + \epsilon z_1^\beta, \qquad z_1'' = \big(|\log z_1| - 1 + \epsilon\beta z_1^{\beta-1}\big) z_1';$$

$$\partial_t v = \omega z_1' z_2, \qquad \partial_{x_1} v = -z_1' z_2, \qquad \partial_{x_1}^2 v = z_1'' z_2, \qquad \partial_{x_2}^2 v = -\left(\frac{\pi}{2a}\right)^2 z_1 z_2.$$

Hence we have

$$\begin{aligned}
&\partial_t\big(v^m\big) - \partial_{x_1}\big(|\partial_{x_1} v|^{p_1-1}\partial_{x_1} v\big) - \partial_{x_2}\big(|\partial_{x_2} v|^{p_2-1}\partial_{x_2} v\big) - v^{\alpha-1}\partial_{x_1} v \\
&\quad = m\omega z_1^{m-1} z_2^m z_1' - p_1\big|z_1' z_2\big|^{p_1-1} z_2 z_1'' - p_2\big|z_1 z_2'\big|^{p_2-1} z_1 z_2'' + z_1^{\alpha-1} z_2^\alpha z_1' \\
&\quad \leqslant m\omega z_1^{m-1} z_2^m z_1' - p_1\big|z_1'\big|^{p_1} z_2^{p_1}\big(|\log z_1| - 1\big) + p_2 z_1^{p_2}\big|z_2'\big|^{p_2-1} z_2\left(\frac{\pi}{2a}\right)^2 + z_1^{\alpha-1} z_2^\alpha z_1' \\
&\quad \leqslant z_1 z_2\big[\big(m\omega z_1^{m-2} z_2^{m-1} + z_1^{\alpha-2} z_2^{\alpha-1}\big)\big(z_1|\log z_1| + \epsilon z_1^\beta\big) - p_1 z_1^{p_1-1} z_2^{p_1-1}|\log z_1|^{p_1}\big(|\log z_1| - 1\big) + C p_2 z_1^{p_2-1}\big] \\
&\quad \leqslant 0
\end{aligned}$$

for $z_1 \in (0, \eta)$, provided that $\eta > 0$ is sufficiently small, where $C = (\frac{\pi}{2a})^{p_2+1}$ as above.

In order to guarantee $0 \leqslant z_1 \leqslant \eta$, we choose $\epsilon > 0$ so small that

$$\zeta_0 = \int_0^\eta \frac{ds}{s|\log s| + \epsilon s^\beta} \geqslant \omega\tau,$$

and consequently, $\eta = z_1(\zeta_0) \geqslant z_1(\omega\tau) \geqslant z_1(\omega t - x_1) \geqslant 0$ in Q_τ^a.

Thus, $v(x_1, x_2, t)$ is a sub-solution of (3.1) in Q_τ^a.

In addition, we have $v \leqslant u$ in $\overline{Q_\tau^a} \setminus Q_\tau^a$. In fact,

$$\begin{aligned}
&v(0, x_2, t) \leqslant \eta \leqslant u(0, x_2, t) \quad \text{for } \big|x_2 - x_2^0\big| \leqslant a,\ 0 \leqslant t \leqslant \tau; \\
&v\big(x_1, x_2^0 \pm a, t\big) = z_1(\omega t - x_1) z_2(\pm a) = 0 \leqslant u\big(x_1, x_2^0 \pm a, t\big); \\
&v(x_1, x_2, 0) = z_1(-x_1) z_2\big(x_2 - x_2^0\big) = 0 \leqslant u(x_1, x_2, 0).
\end{aligned}$$

Therefore, by the comparison principle, $v \leqslant u$ in Q_τ^a. In particular,

$$u(x_1^0, x_2, t_0) \geqslant v(x_1^0, x_2, t_0) = z_1(\omega t_0 - x_1^0) z_2(x_2 - x_2^0) > 0,$$

since $\omega t_0 - x_1^0 = x_1^0 > 0$, and $|x_2 - x_2^0| < a$. From the arbitrariness of $(x_1^0, x_2, t_0) \in Q_\tau^a$, the conclusion follows, and the theorem is then proved.

(b) The proof is almost the same as part (a). □

3.2. Finite speed propagation

Theorem 3.2. *Let $u(x_1, x_2, t)$ be a solution of (3.1) in Q_T.*

(a) *Assume that $0 < m < p_2$. If $u(0, x_2, t) = 0$ for $x_2 > 0$ sufficiently large, $0 < t \leqslant T$, and initial support $\operatorname{supp} u(x_1, x_2, 0)$ is bounded in x_2-direction, then there exist $r(t) > 0$ for all $t \in (0, T]$, such that*

$$u(x_1, x_2, t) = 0 \quad \text{for } x_1 \geqslant 0,\ x_2 \geqslant r(t),\ 0 < t \leqslant T.$$

(b) *Assume that $0 < m < p_1$. If $u(x_1, 0, t) = 0$ for $x_1 > 0$ sufficiently large, $0 < t \leqslant T$, and initial support $\operatorname{supp} u(x_1, x_2, 0)$ is bounded in x_1-direction, then there exist $r(t) > 0$ for all $t \in (0, T]$, such that*

$$u(x_1, x_2, t) = 0 \quad \text{for } x_1 \geqslant r(t),\ x_2 \geqslant 0,\ 0 < t \leqslant T.$$

Proof. (a) Let $u(x_1, x_2, t)$ be a solution of (3.1) in Q_T, satisfying with some $c > 0$,

$$u(0, x_2, t) = 0 \quad \text{for } x_2 \geqslant c,\ 0 \leqslant x_1 < \infty,$$

and

$$u(x_1, x_2, 0) = 0 \quad \text{for } x_2 \geqslant c,\ 0 \leqslant t \leqslant T.$$

Assume that $M = \max u < \infty$ and consider $w(x_1, x_2, t) = z(\omega t - x_2 + c + \zeta_0)$ in Q_T, where function $z(\zeta)$, as well as ζ_0, is defined by

$$\int_0^{z(\zeta)} \frac{ds}{s^{m/p_2}} = \zeta_+, \qquad \zeta_0 := \int_0^M \frac{ds}{s^{m/p_2}}.$$

Hence $z' = z^{m/p_2}$, $z'' = \frac{m}{p_2} z^{\frac{m}{p_2}-1} z'$, and

$$\begin{aligned}
&\partial_t(w^m) - \partial_{x_1}\left(|\partial_{x_1} w|^{p_1-1}\partial_{x_1} w\right) - \partial_{x_2}\left(|\partial_{x_2} w|^{p_2-1}\partial_{x_2} w\right) - w^{\alpha-1}\partial_{x_1} w \\
&\quad = \omega m z^{m-1} z' + \partial_{x_2}\left(|-z'|^{p_2-1} z'\right) \\
&\quad = (\omega - 1) m z^{m-1} z' \geqslant 0,
\end{aligned}$$

provided that we choose $\omega \geqslant 1$. Therefore $w(x_1, x_2, t)$ is a super-solution of (3.1) in Q_T. Noting that if $x_2 \leqslant c$, then

$$w \geqslant z(\omega t + \zeta_0) \geqslant z(\zeta_0) = M \geqslant u;$$

and if $x_2 \geqslant c$, then

$$u|_{t=0} = 0 \leqslant w|_{t=0}, \qquad u|_{x_1=0} = 0 \leqslant w|_{x_1=0}.$$

These imply that $w \geqslant u$ on $\overline{Q}_T \setminus Q_T$. By the comparison principle $w \geqslant u$ on $\overline{Q}_T$. Particularly,

$$u(x_1, x_2, t) = 0 \quad \text{for } x_1 \geqslant 0,\ x_2 \geqslant \omega t + c + \zeta_0,\ 0 \leqslant t \leqslant T.$$

The proof is then completed.

(b) The proof is the same as that of part (a). □

3.3. Space localization

Theorem 3.3. *Let $\alpha < p_1$, and $u(x_1, x_2, t)$ be a solution of (3.1) in Q_T with initial support bounded in x_1-direction, and $u(x_1, 0, t) = 0$ for $x_1 > 0$ large enough. Then there exists $r_0 > 0$ so that*

$$u(x_1, x_2, t) = 0 \quad \text{for } x_1 \geqslant r_0,\ x_2 \geqslant 0,\ 0 \leqslant t \leqslant T.$$

Here r_0 is independent of T. This is called localization of perturbation.

Proof. Let $M = \max u < \infty$ and $a > 0$ be such that

$$u(x_1, x_2, 0) = 0 \quad \text{for } x_1 \geqslant a,\ 0 \leqslant x_2 < \infty,$$

$$u(x_1, 0, t) = 0 \quad \text{for } x_1 \geqslant a,\ 0 \leqslant t \leqslant T.$$

Then we consider $w(x_1, x_2, t) = z(-x_1 + a + \zeta_0)$ in Q_T, where $z(\zeta)$ and ζ_0 are defined by

$$\int_0^{z(\zeta)} \frac{ds}{s^{\alpha/p_1}} = \zeta_+, \qquad \zeta_0 := \int_0^M \frac{ds}{s^{\alpha/p_1}}.$$

Thus $z' = z^{\alpha/p_1}$, $z'' = \frac{\alpha}{p_1} z^{\frac{\alpha}{p_1}-1} z'$, and

$$\begin{aligned} &\partial_t\big(w^m\big) - \partial_{x_1}\big(|\partial_{x_1} w|^{p_1-1}\partial_{x_1} w\big) - \partial_{x_2}\big(|\partial_{x_2} w|^{p_2-1}\partial_{x_2} w\big) - w^{\alpha-1}\partial_{x_1} w \\ &\quad = -\partial_{x_1}\big(|-z'|^{p_1-1}(-z')\big) + z^{\alpha-1}z' \\ &\quad = (-\alpha+1)z^{\alpha-1}z' \geqslant 0. \end{aligned}$$

Therefore $w(x_1, x_2, t)$ is a super-solution of (3.1) in Q_T.

Note that if $x_1 \leqslant a$, then $w \geqslant z(\zeta_0) = M \geqslant u$; and if $x_1 \geqslant a$, then $u|_{t=0} = 0 \leqslant w|_{t=0}$, $u|_{x_2=0} = 0 \leqslant w|_{x_2=0}$. These imply that $w \geqslant u$ on $\overline{Q}_T \setminus Q_T$. By the comparison principle $w \geqslant u$ on $\overline{Q}_T$. Particularly,

$$u(x_1, x_2, t) = 0 \quad \text{for } x_1 \geqslant a + \zeta_0,\ x_2 \geqslant 0,\ 0 \leqslant t \leqslant T.$$

The proof is then completed. □

4. Conclusion and open problem

We summarize the preceding results as follows.

First of all, it is concerned with the equation

$$\partial_t\big(u^m\big) - \partial_{x_1}\big(|\partial_{x_1} u|^{p_1-1}\partial_{x_1} u\big) - \partial_{x_2}\big(|\partial_{x_2} u|^{p_2-1}\partial_{x_2} u\big) + u^{\alpha-1}\partial_{x_1} u = 0, \tag{4.1}$$

we have

- If $0 < p_1 \leqslant \min\{p_2, m, 1\}$, then Eq. (4.1) admits x_1-ISP, and if $0 < p_2 \leqslant \min\{p_1, m, 1\}$, $p_2 + 1 \leqslant \alpha$, the equation admits x_2-ISP;
- If $0 < m < p_1$, $m \leqslant \alpha$, then Eq. (4.1) admits x_1-FSP and if $0 < m < p_2$ then (4.1) admits x_2-FSP;
- If $0 < \alpha < m < p_2$, then Eq. (4.1) admits x_1-ISP and x_2-FSP, i.e., anisotropic propagation occurs.

Another anisotropic propagation may appear as follows.

- If $0 < p_2 \leqslant m < p_1$, $p_2 \leqslant 1$, $m \leqslant \alpha$, $p_2 + 1 \leqslant \alpha$, then Eq. (4.1) admits x_1-FSP and x_2-ISP.

Next we deal with the equation

$$\partial_t\big(u^m\big) - \partial_{x_1}\big(|\partial_{x_1} u|^{p_1-1}\partial_{x_1} u\big) - \partial_{x_2}\big(|\partial_{x_2} u|^{p_2-1}\partial_{x_2} u\big) - u^{\alpha-1}\partial_{x_1} u = 0, \tag{4.2}$$

and summarize conclusions as:

- If $0 < m < p_i$, then Eq. (4.2) admits x_i-FSP, $i = 1, 2$;
- If $0 < p_1 \leqslant \min\{p_2, m, \alpha, 1\}$, then Eq. (4.2) admits x_1-ISP, and if $0 < p_2 \leqslant \min\{p_1, m, 1\}$, $p_2 + 1 \leqslant \alpha$, then Eq. (4.2) admits x_2-ISP;
- If $0 < \alpha < p_1$, then Eq. (4.2) admits x_1-FSP.

These plainly imply possibility of anisotropic propagation.

Particularly, applying the above conclusions to the equation without convection term, i.e.,

$$\partial_t\big(u^m\big) - \partial_{x_1}\big(|\partial_{x_1}u|^{p_1-1}\partial_{x_1}u\big) - \partial_{x_2}\big(|\partial_{x_2}u|^{p_2-1}\partial_{x_2}u\big) = 0, \tag{4.3}$$

we have

- If $0 < m < p_i$, then Eq. (4.3) admits x_i-FSP, $i = 1, 2$;
- If $0 < p_1 \leqslant \min\{p_2, m, 1\}$, then Eq. (4.3) admits x_1-ISP, and if $0 < p_2 \leqslant \min\{p_1, m, 1\}$, then Eq. (4.3) admits x_2-ISP.

Anisotropic propagation may also happen. For instance,

- If $0 < p_1 \leqslant m < p_2$, $p_1 \leqslant 1$, then the equation admits x_1-ISP and x_2-FSP.

Remark 4.1. In the case that $p_2 \leqslant m$ and $0 < \alpha < p_2 + 1$, we do not know whether Eqs. (4.1) and (4.2) admit x_2-ISP or not.

Remark 4.2. In the case that $1 < p_1 \leqslant m < p_2$, we do not know the propagation property in x_1 direction for Eq. (4.3).

Remark 4.3. When $p_1 < p_2 \leqslant m$ holds, we believe that Eq. (4.3) admits ISP (at least x_1-ISP), but have not proven it yet.

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